XXVI. *A new Method of finding the equal Roots of an Equation, by Diviſion. By the Rev.* John Hellins, *Curate of* Conſtantine, *in* Cornwall; *communicated by* Nevil Maſkelyne, *D. D. F. R. S. and Aſtronomer Royal.*

Read June 20, 1782.

THE following theorems are a production of juvenile years. They were invented about twelve years ago, when algebra was my favourite ſtudy; and one of them (the firſt) was publiſhed as a ſpecimen of this method of extracting the equal roots of an equation about ten years ago. Since that time my avocations have left me but very little leiſure for improving any invention of this kind. Theſe theorems, then, are in their crude ſtate; however, ſuch as they are, I flatter myſelf, they will afford an eaſier ſolution of equations that have equal roots than is generally known, and be acceptable to the ingenious algebraiſt.

THEOREM I.

If the cubic equation $x^3 - px^2 + qx - r = 0$ *has two equal roots, each of them will be* $(x) = \frac{pq - 9r}{2pp - 6q}$.

DEMONSTRATION.

Call the three roots a, a, and b; then, by the compoſition of equations we ſhall have $x^3 \left.\begin{matrix}-2a\\-b\end{matrix}\right\} x^2 \left.\begin{matrix}+aa\\+2ab\end{matrix}\right\} x - aab = 0$, where

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$2a+b=p$, $aa+2ab=q$, and $aab=r$; which values being written in our theorem, we have $x\ \left(=\frac{pq-9r}{2pp-6q}\right)=$ $\frac{2aaa+4aab+aab+2abb-9aab}{8aa+8ab+2bb-6aa-12ab}=\frac{2aaa-4aab+2abb}{2aa-4ab+2bb}=a$. *Q. E. D.*

EXAMPLE I.

If the equation $x^3+5x^2-32x+36=0$ has two equal roots, it is propoſed to find them by the above theorem.

Here $p=-5$, $q=-32$, and $r=-36$; theſe values being written in the theorem, we have $\frac{-5\times-32-9\times-36}{2\times25-6\times-32}=\frac{160+324}{50+192}=\frac{484}{242}=2$, which being written for x, the equation becomes $8+20-64+36$, which is evidently $=0$; conſequently 2 and 2 are roots of it.

Otherwiſe, 2, the value of x given by the theorem, being written for it in the quadratic equation $3x^2+10x-32=0$, the reſult is $12+20-32=0$.

Or, dividing the given cubic by the quadratic $\overline{x-2}|^2$, we have $x^2-4x+4)\,x^3+5x^2-32x+36\,(x+9$; therefore the three roots are 2, 2, and -9.

EXAMPLE II.

Given $x^3+\frac{10}{7}x^2-\frac{4000}{9261}=0$, an equation which has equal roots, to find them.

Here $q=0$, and the theorem gives $\frac{-36000\times49}{200\times9261}=\frac{-20}{21}$, which value being written for x the equation vaniſhes.

THEOREM II.

If the biquadratic equation $x^4 - px^3 + qx^2 - rx + s = 0$ *has two equal roots, make* $A = \frac{12r - 2pq}{3pp - 8q}$, $B = \frac{pr - 16s}{3pp - 8q}$, $C = \frac{4B - 2q}{4A + 3p}$, *and* $D = \frac{r}{4A + 3p}$, *and you will have* $x = \frac{D - B}{A - C}$.

A ſynthetical demonſtration of this theorem would be very long: the INVESTIGATION is as follows.

It has been demonſtrated by the writers on algebra, that, if a biquadratic equation, as $x^4 - px^3 + qx^2 - rx + s = 0$, has two equal roots, one of them may be had from the equation $4x^3 - 3px^2 + 2qx - r = 0$. Multiply this equation by x, and the original one by 4, and take the difference of the two, which will be $px^3 - 2qx^2 + 3rx - 4s = 0$. Again, if this equation be multiplied by 4, and the other cubic by p, and their difference taken, we ſhall have $\overline{3pp - 8q} \times x^2 + \overline{12r - 2pq} \times x + pr - 16s = 0$, or $x^2 + \frac{12r - 2pq}{3pp - 8q} x + \frac{pr - 16s}{3pp - 8q} = 0$, or $x^2 + Ax + B = 0$, putting A and B for the known quantities in the ſecond and third terms. Now multiply this equation by $4x$, and take the firſt cubic from it, and we ſhall have $\overline{4A + 3p} \times x^2 + \overline{4B - 2q} \times x + r = 0$, which being divided by $4A + 3p$, and C and D put equal to $\frac{4B - 2q}{4A + 3p}$ and $\frac{r}{4A + 3p}$ reſpectively, gives $x^2 + Cx + D = 0$; and this equation being taken from the other quadratic, there remains $\overline{A - C} \times x + B - D = 0$; conſequently $x = \frac{D - B}{A - C}$. *Q. E. I.*

COROLLARY I. From the above inveſtigation it appears, that one of the equal roots may alſo be obtained from either of theſe two quadratic equations, of which the firſt ſeems moſt eligible, as the co-efficients of it are leſs complex than thoſe of the other:

$$\overline{3pp-8q}\times x^2+\overline{12r-2pq}\times x+pr-16s=0,$$

and $\overline{4A+3p}\times x^2+\overline{4B-2q}\times x+r=0$. And theſe, when $p=0$, become $-8qx^2+12rx-16s=0$,

$$\text{and } -\frac{48r}{8q}x^2+\overline{\frac{64s}{8q}-2q}\times x+r=0,$$

$$\text{or } x^2-\frac{3r}{2q}x+\frac{2s}{q}=0,$$

$$\text{and } x^2+\frac{qq-4s}{3r}x-\frac{q}{6}=0.$$

COROL. 2. If both p and q vaniſh, then, from either of the quadratics we get $x=\frac{4s}{3r}$, perfectly agreeing with the cubic $px^3-2qx^2+3rx-4s=0$, which, when p and q vaniſh, becomes $3rx-4s=0$. And this equation is of uſe; becauſe, in this caſe, the theorem fails, one of the diviſors being $=0$.

COROL. 3. From the equation $4x^3-3px^2+2qx-r=0$, which, when p and q vaniſh, becomes $4x^3-r=0$, we alſo get $x=\sqrt[3]{\frac{r}{4}}$, another expreſſion of the ſame value of x.

COROL. 4. When $r=0$, $D=0$, and from the equation $x^2+Cx+D=0$, we have $x=-C$.

EXAMPLE I.

If the equation $x^4 * -9x^2+4x+12=0$ has equal roots, it is propoſed to find them.

Here $p=0$, $q=-9$, $r=-4$, and $s=12$; and

$$\text{A becomes} = \frac{12\times-4}{-8\times-9} = \frac{-2}{3},$$

$$\text{B} \quad . \quad . \quad = \frac{-16\times-12}{-8\times-9} = \frac{-8}{3},$$

$$\text{C} \quad . \quad . \quad = \frac{4\times\frac{-8}{3}+18}{4\times\frac{-2}{3}} = \frac{-11}{4},$$

$$\text{D} \quad . \quad . \quad = \frac{-4}{\frac{-8}{3}} = \frac{3}{2},$$

$$\text{and } \frac{\text{D}-\text{B}}{\text{A}-\text{C}} = \frac{\frac{3}{2}+\frac{8}{3}}{\frac{-2}{3}+\frac{11}{4}} = \frac{18+32}{-8+33} = \frac{50}{25} = 2,$$

which being written for x, the equation becomes $16-36+8+12=0$; therefore 2 is one of the roots.

The ſame value of x may be diſcovered from either of the quadratic equations mentioned in corollary 1. The proper values of the co-efficients being written in the firſt of them, it becomes $x^2-\frac{2}{3}x-\frac{8}{3}=0$, where one value of x is $\frac{1+\sqrt{25}}{3}=2$. The other quadratic becomes $x^2-\frac{11}{4}x+\frac{3}{2}=0$, one of whoſe roots is $\frac{11+\sqrt{25}}{8}=2$.

EXAMPLE II.

It being known that the equation $x^4-x^3-7x^2+13x-6=0$ has two equal roots, to find them.

Here $p=1$, $q=-7$, $r=-13$, and $s=-6$; and $\text{A}=\frac{-142}{59}$, $\text{B}=\frac{83}{59}$, $\text{C}=\frac{-1158}{391}$, $\text{D}=\frac{767}{391}$, $\text{D}-\text{B}=\frac{12800}{23009}$, $\text{A}-\text{C}=\frac{12800}{23069}$, and $\frac{\text{D}-\text{B}}{\text{A}-\text{C}}=\frac{12800}{12800}=1$, one of the roots ſought.

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The ſame value of x may be found from either of the two general quadratic equations given in corollary 1. From the firſt of them we get one value of $x = \frac{71 - \sqrt{144}}{59} = 1$. And from the other, one value $= \frac{579 - \sqrt{35344}}{391}$, which is alſo $= 1$.

EXAMPLE III.

Given the equation $x^4 - \frac{1}{2}x + \frac{3}{16} = 0$, in which two values of x are equal to each other, to find them.

By corollary 2. we have $x = \frac{4 \times 3}{16} \div \frac{3 \times 1}{2} = \frac{8}{16} = \frac{1}{2}$. By corol. 3. x is $= \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$.

THEOREM III.

If the ſurſolid equation $x^5 - px^4 + qx^3 - rx^2 + sx - t = 0$ *has two roots equal to each other, and you make* $A = \frac{15r - 3pq}{4pp - 10q}$, $B = \frac{2pr - 20s}{4pp - 10q}$, $C = \frac{25t - ps}{4pp - 10q}$, $D = \frac{5B - 3q}{5A + 4p}$, $E = \frac{5C + 2r}{5A + 4p}$, $F = \frac{s}{5A + 4p}$, $G = \frac{B - E}{A - D}$, $H = \frac{F + C}{A - D}$, $I = \frac{B - H}{A - G}$, *and* $K = \frac{C}{A - G}$, *then ſhall one of the equal values of* x *be* $= \frac{H - K}{I - G}$.

The inveſtigation of this theorem being altogether ſimilar to that of the laſt, it is unneceſſary to give it here.

The difference of equations being taken as in the inveſtigation of theorem II. it will appear, that one of the equal roots may alſo be had from any one of the following five equations, of which ſometimes one, ſometimes another, will be the moſt eligible.

1. $5x^4 - 4px^3 + 3qx^2 - 2rx + s = 0.$
2. $px^4 - 2qx^3 + 3rx^2 - 4sx + 5t = 0.$
3. $x^3 + Ax^2 + Bx + C = 0.$
4. $x^3 + Dx^2 + Ex - F = 0.$
5. $x^2 + Gx + H = 0.$

It is obvious, that, when p vaniſhes, the work will be conſiderably ſhortened; and when both p and q are wanting, though the above *formula* fails, yet the equal root may be eaſily obtained from the equation $px^4 - 2qx^3 + 3rx^2 - 4sx + st = 0$, which in that caſe becomes $3rx^2 - 4sx + 5t = 0$. Whenever s is wanting, F, in the ſecond cubic above, will be $= 0$, and conſequently x may be found from the quadratic equation $x^2 + Dx + E = 0$. But in any of theſe caſes the equal root may be found by diviſion. However, the operation probably will not, in general, be ſo ſhort as extracting the root of the quadratic; I will therefore haſten to give an example or two of the uſe of the theorem.

EXAMPLE I.

Given $x^5 + x^3 - x^2 + 0{\cdot}09433 = 0$, to find x, two values of it being equal to each other.

Here $p = 0$, $q = 1$, $r = 1$, $s = 0$, $t = -0{\cdot}09433$, and we get

$A = -1{\cdot}5$ $\quad F = 0$

$B = 0$ $\quad G = -0{\cdot}2231$

$C = +0{\cdot}2358$ $\quad H = -0{\cdot}1241$

$D = +0{\cdot}4$ $\quad I = -0{\cdot}0972$

$E = -0{\cdot}4238$ $\quad K = -0{\cdot}185$

$$\text{and } x = \frac{H - K}{I - G} = 0.48.$$

The proper values of the co-efficients being written in the five equations before mentioned, and ſome of them divided by the

the co-efficient of the higheſt power of x, we have theſe four equations, in each of which one value of x is one of the equal ones ſought :

$$x^3 \ast + 0{\cdot}6x - 0{\cdot}4 = 0.$$
$$x^3 - 1{\cdot}5x^2 \ast + 0{\cdot}2358 = 0.$$
$$x^2 + 0{\cdot}4x - 0{\cdot}4238 = 0.$$
$$x^2 - 0{\cdot}2231x - 0{\cdot}1241 = 0.$$

Now the moſt eligible equation is the quadratic $x^2 + 0{\cdot}4x - 0{\cdot}4238 = 0$, whoſe affirmative root is $\sqrt{0{\cdot}4638} - 0{\cdot}2 = 0{\cdot}4811$, agreeing with the value of x found above, but true to two places lower in the decimal.

EXAMPLE II.

To find the two equal values of x in the equation $64x^5 - 20x^2 + 3 = 0$.

The given equation being divided by 64, we have $x^5 - 0{\cdot}3125x^2 + 0{\cdot}046875 = 0$; and then, from the firſt of the five equations given above, we get $5x^4 - 0{\cdot}625x = 0$, and $x = \sqrt[3]{0{\cdot}125} = 0{\cdot}5$. But from the ſecond of the equations juſt mentioned, we have $0{\cdot}9375x^2 - 0{\cdot}234375 = 0$, or $x^2 = \frac{0{\cdot}234375}{0{\cdot}9375} = 0{\cdot}25$, and $x = \sqrt{0{\cdot}25} = 0{\cdot}5$.

From the foregoing few pages it is evident, that rules may be made for finding the equal roots of equations of more than five dimenſions by diviſion ; but the operations by them will, in moſt caſes, be long and tedious. It is obvious, however, that ſuch equations may be depreſſed to any dimenſion the algebraiſt pleaſes.

It has indeed been ſuppoſed, that the number of equations that have equal roots is but ſmall, and, conſequently, that the chief uſe of the rules for finding their roots is to get limits and approximations to the roots of equations in general. That uſe, it muſt be allowed, were it the only one, is ſufficient to pay for inveſtigating them. But if the equations that have equal roots ſhould hereafter be found not ſo few as has been generally received, then the uſe of the above theorems will become more extenſive.

I beg leave to add, that this ſhort eſſay is but a ſmall part of a work, in which, if I ſhould ever have leiſure to put a finiſhing hand to it, ſomething more on this ſubject may very probably appear. In the mean while, I hope, this little piece will be candidly received by thoſe who have more leiſure and better abilities for ſtudies of this kind.

Conſtantine,
February 9, 1782.